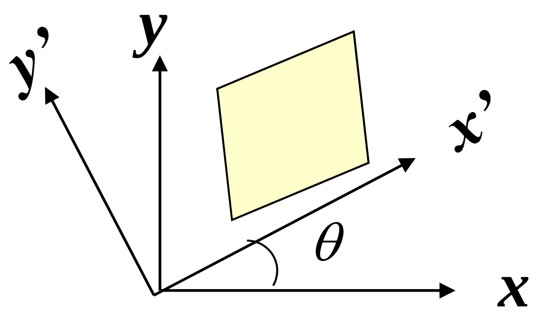
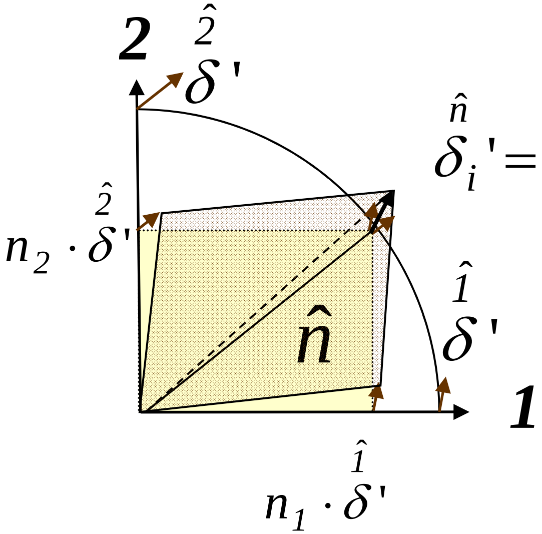
Three dimensional strain analysis



$$\varepsilon_{x'} = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\cdot \cos 2\theta + \frac{\gamma_{xy}}{2}\cdot \sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\cdot \sin 2\theta + \frac{\gamma_{xy}}{2}\cdot \cos 2\theta$$



$$\overset{\hat{n}}{\delta}_i' = n_j \cdot \varepsilon_{ji}'$$

$$\varepsilon_{ij}' = \frac{1}{2}(\varepsilon_{ij}' + \varepsilon_{ji}') + \frac{1}{2}(\varepsilon_{ij}' - \varepsilon_{ji}')$$

$$= \varepsilon_{ij} + w_{ij}$$

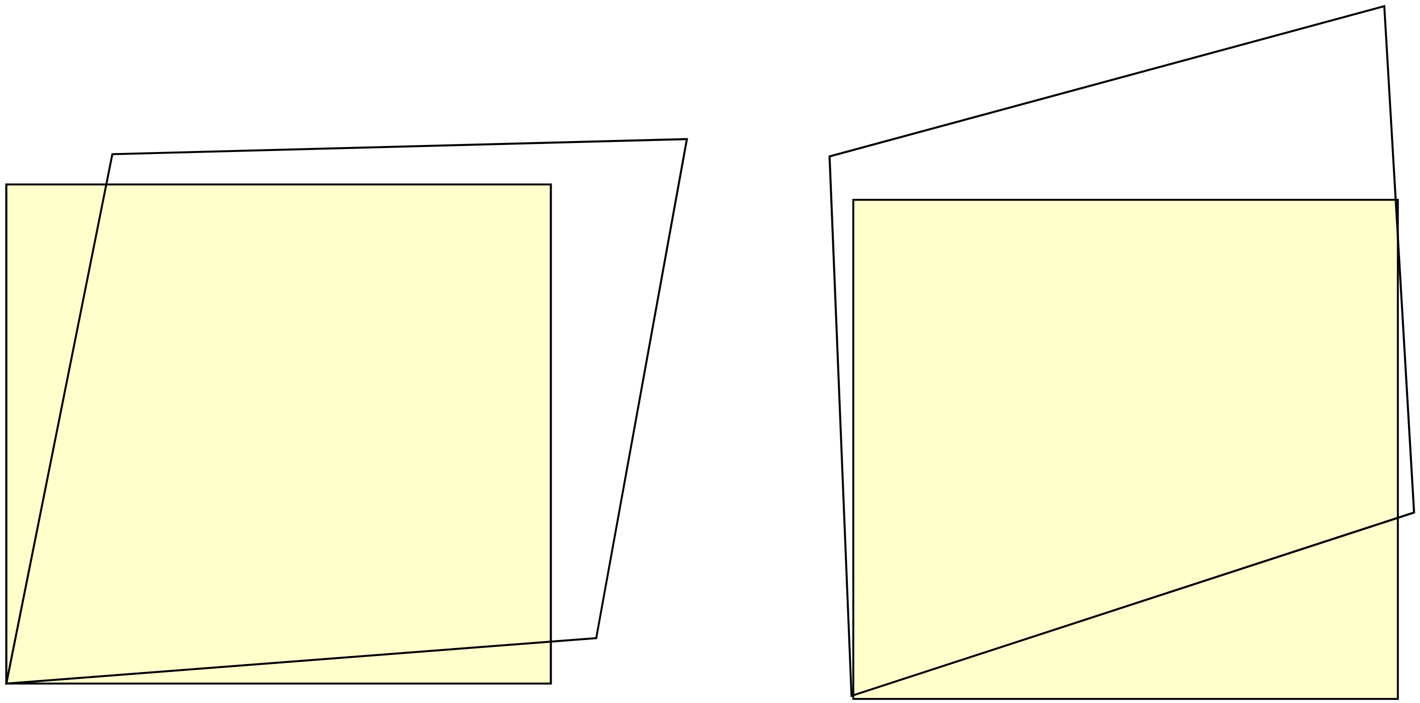
$$\overset{\hat{n}}{\delta}_i' = n_j \cdot \varepsilon_{ji} + n_j \cdot w_{ji}$$

$$= \varepsilon_{ij} \cdot n_j - w_{ij} \cdot n_j$$

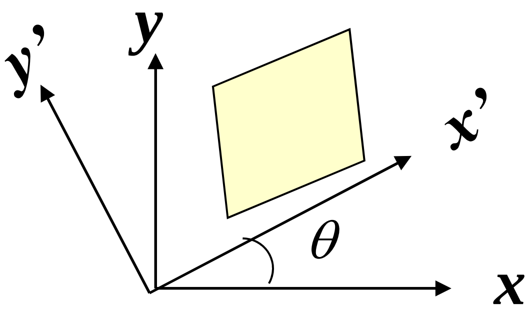
$$= \overset{\hat{n}}{\delta}_i + \overset{\hat{n}}{\Omega}_i$$

$$\overset{\hat{n}}{\delta}_i = \varepsilon_{ij} \cdot n_j$$

Pure deformation ***Rigid body rotation***



Three dimensional strain analysis

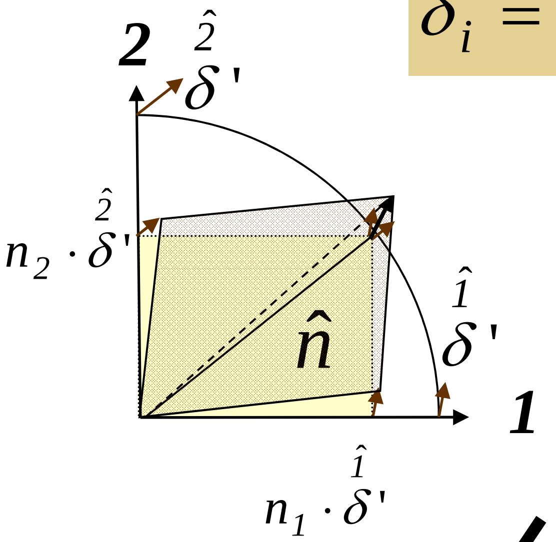


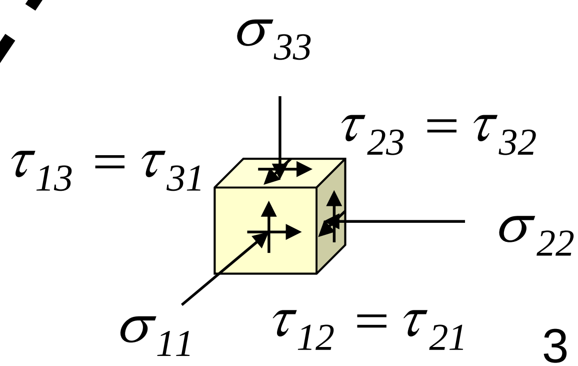
$$\varepsilon_{x'} = (\frac{\varepsilon_x + \varepsilon_y}{2}) + (\frac{\varepsilon_x - \varepsilon_y}{2}) \cdot \cos 2\theta + \frac{\gamma_{xy}}{2} \cdot \sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -(\frac{\varepsilon_x - \varepsilon_y}{2}) \cdot \sin 2\theta + \frac{\gamma_{xy}}{2} \cdot \cos 2\theta$$

Strain analysis

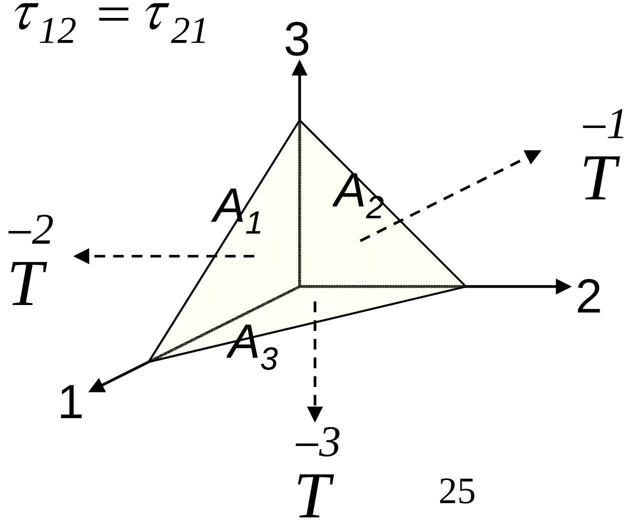
$$\overset{\hat{n}}{\delta}_i = \varepsilon_{ij} \cdot n_j$$



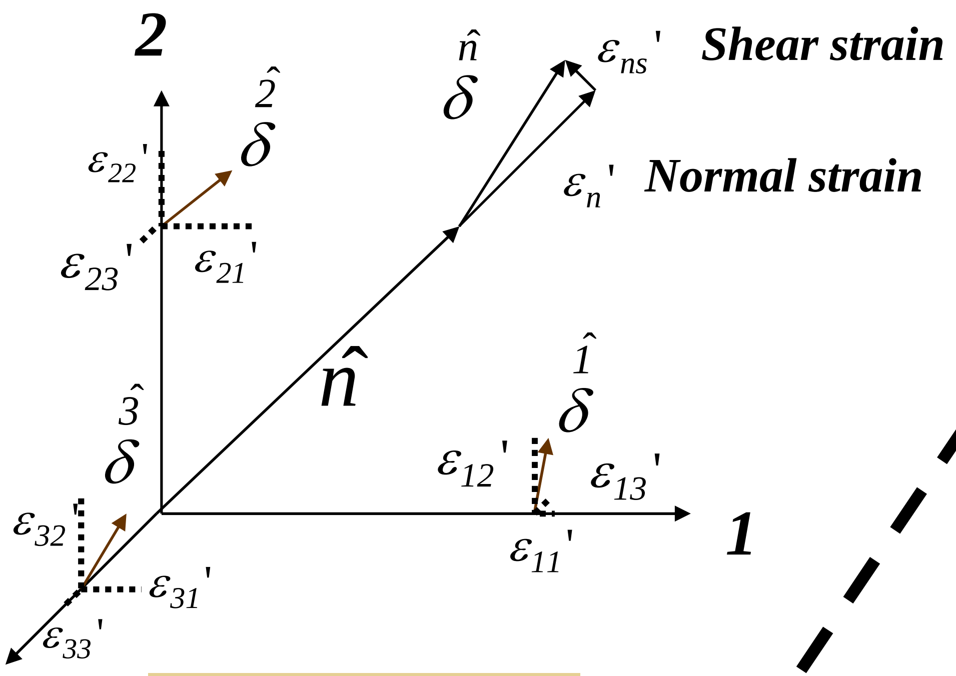


Stress analysis

$$\overset{\hat{n}}{T}_i = \sigma_{ji} \cdot n_j = \sigma_{ij} \cdot n_j$$



Three dimensional strain analysis



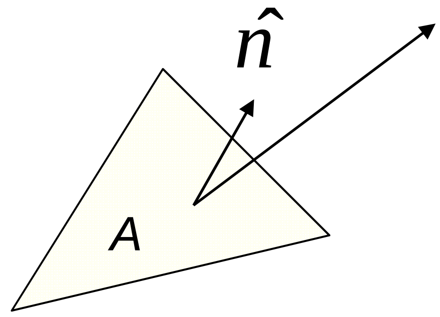
$$\varepsilon_n = \overset{\hat{n}}{\delta}_i \cdot n_i = \varepsilon_{ij} \cdot n_i \cdot n_j$$

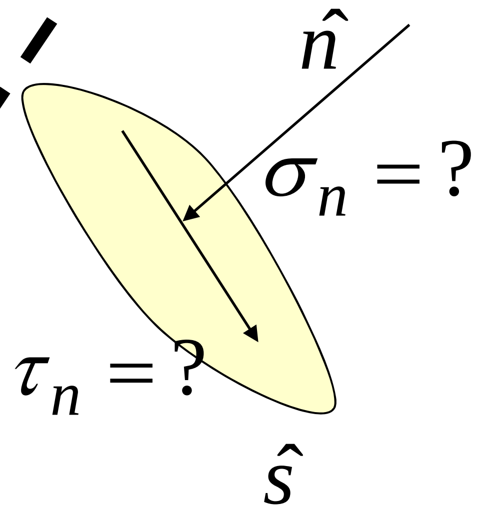
$$\varepsilon_{ns} = \overset{\hat{n}}{\delta}_i \cdot s_i = \varepsilon_{ij} \cdot n_i \cdot s_j$$

Strain analysis

Stress analysis

$$\overset{\hat{n}}{T}_i = \sigma_{ji} \cdot n_j = \sigma_{ij} \cdot n_j$$

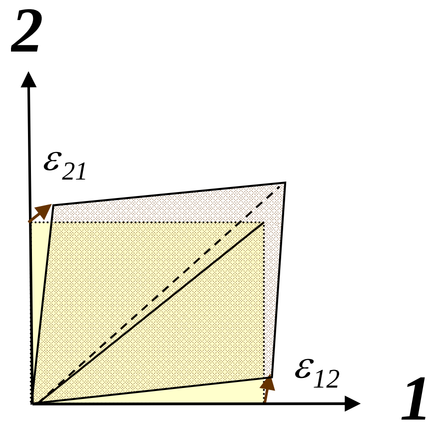




$$\sigma_n = \overset{\hat{n}}{T}_i \cdot n_i = \sigma_{ij} \cdot n_i \cdot n_j$$

$$\tau_n = \overset{\hat{n}}{T}_i \cdot s_i = \sigma_{ij} \cdot n_i \cdot s_j$$

Engineering shear strain $\gamma_{12} = \varepsilon_{12} + \varepsilon_{21} = 2 \cdot \varepsilon_{12}$



Three dimensional strain analysis

Principal strain and principal direction

$$\varepsilon^3 - I_1'\cdot\varepsilon^2 + I_2'\cdot\varepsilon - I_3' = 0$$

3 solutions of strain represent 3 principal strain $\varepsilon_1, \varepsilon_2, \varepsilon_3$

Correspond orthogonal directions are principal directions

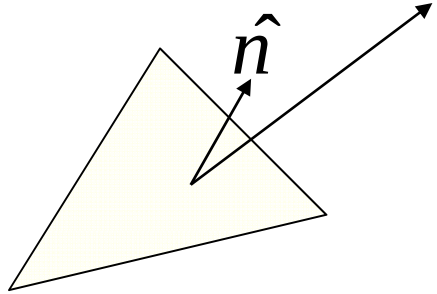
$$\varepsilon = \varepsilon_1$$

$$\begin{bmatrix} \varepsilon_{11}-\varepsilon_1 & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22}-\varepsilon_1 & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}-\varepsilon_1 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad n_1^{(1)}, n_2^{(1)}, n_3^{(1)}$$

The invariants of strain tensor I_1', I_2', I_3'

$$\overset{\hat{n}}{T} = \overset{1}{T}\cdot n_1 + \overset{2}{T}\cdot n_2 + \overset{3}{T}\cdot n_3$$

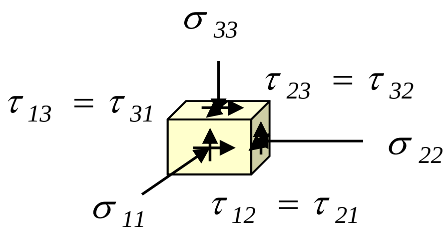


$$\overset{\hat{n}}{T}_i = \hat{\sigma}_n = \sigma\cdot n_i \qquad \tau_n = 0$$

$$\overset{\hat{n}}{T}_i = \sigma\cdot n_i \qquad \overset{\hat{n}}{T}_i = \sigma_{ij}\cdot n_j \quad \Rightarrow \quad \sigma_{ij}\cdot n_j = \sigma\cdot n_i$$

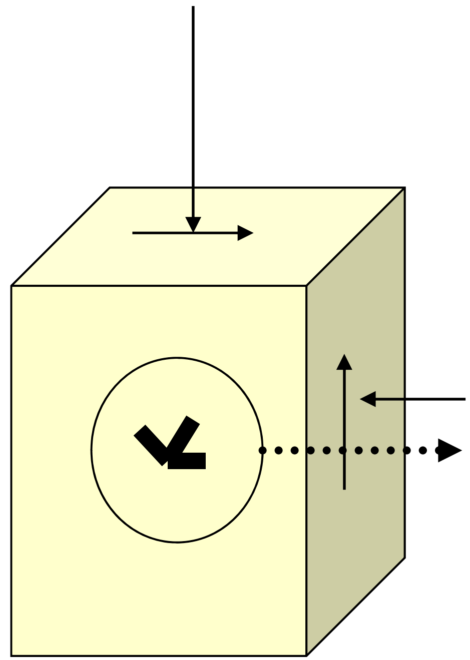
$$\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

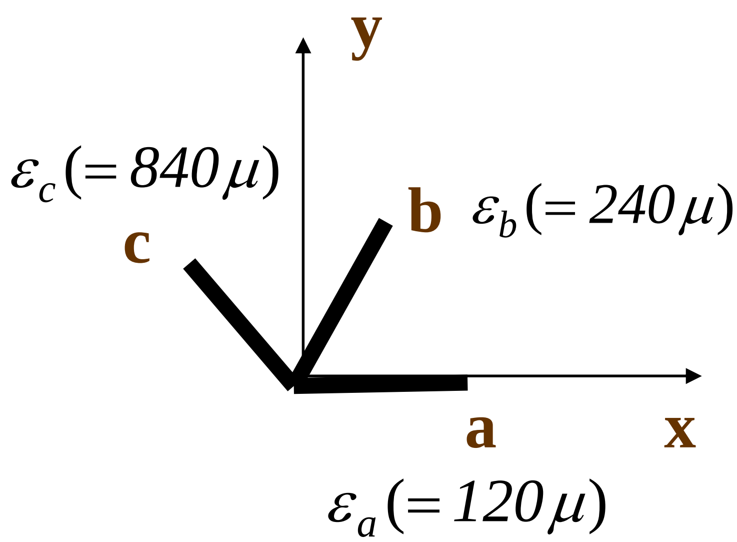
$$\begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = 0$$



剪應力為０之平面為主應力面
該面上作用之正向應力為主應力

Strain gage rosette



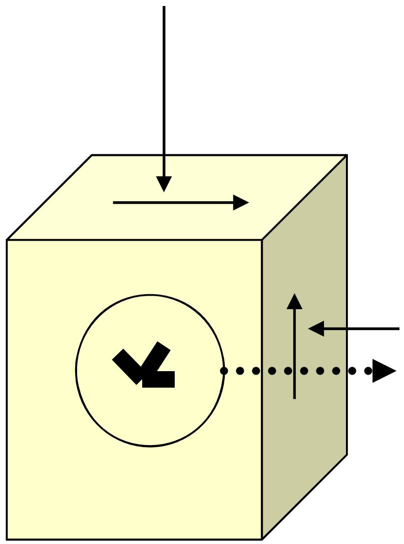


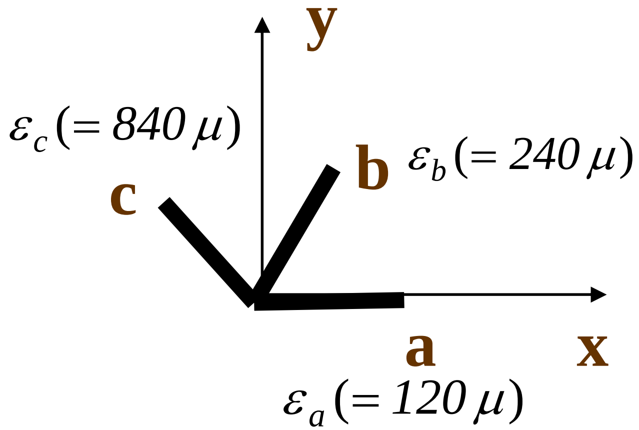
Strain gage a,b,c 分別與水平夾 $\theta_a, \theta_b, \theta_c (= 0^o, 60^o, 120^o)$

How to determine the two dimensional strain tensor $\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$ **?**

三個點決定一個圓方程式！

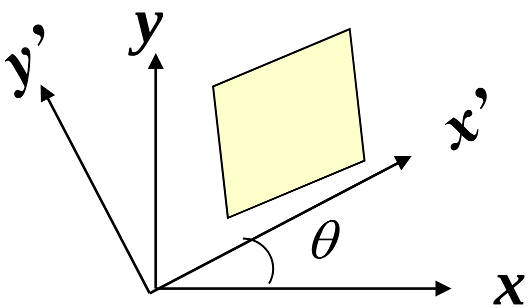
Strain gage rosette





Strain gage a,b,c
分別與水平夾

$$\theta_a,\theta_b,\theta_c(=0^o,60^o,120^o)$$



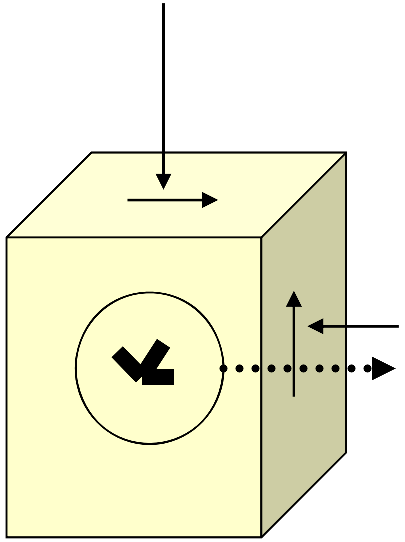
$$\varepsilon(\theta)=\varepsilon_x\cdot\cos^2\theta+\varepsilon_y\cdot\sin^2\theta+\gamma_{xy}\cdot\sin\theta\cdot\cos\theta$$

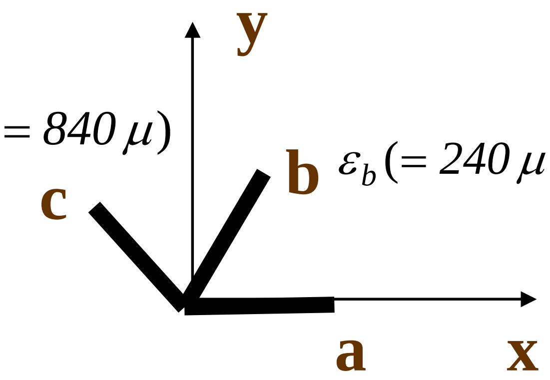
$$=\varepsilon_x\cdot(\frac{1+\cos 2\theta}{2})+\varepsilon_y\cdot(\frac{1-\cos 2\theta}{2})+\frac{1}{2}\cdot\gamma_{xy}\cdot\sin 2\theta$$

$$\varepsilon(\theta)=\varepsilon_{x'}=\begin{pmatrix}\cos^2\theta & \sin^2\theta & \frac{1}{2}\cdot\sin 2\theta\end{pmatrix}\cdot\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{pmatrix}$$

三個未知數，三個方程式！

Strain gage rosette





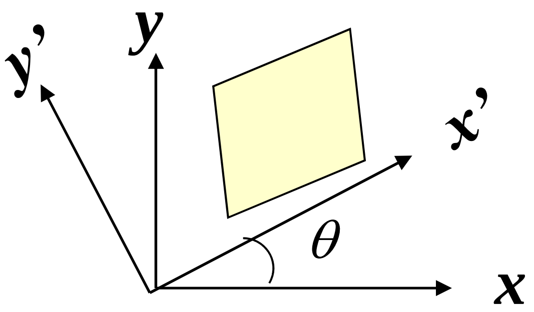
$\varepsilon_c (= 840\,\mu)$

$\varepsilon_b (= 240\,\mu)$

$\varepsilon_a (= 120\,\mu)$

Strain gage a,b,c
分別與水平夾

$$\theta_a, \theta_b, \theta_c (= 0^o, 60^o, 120^o)$$



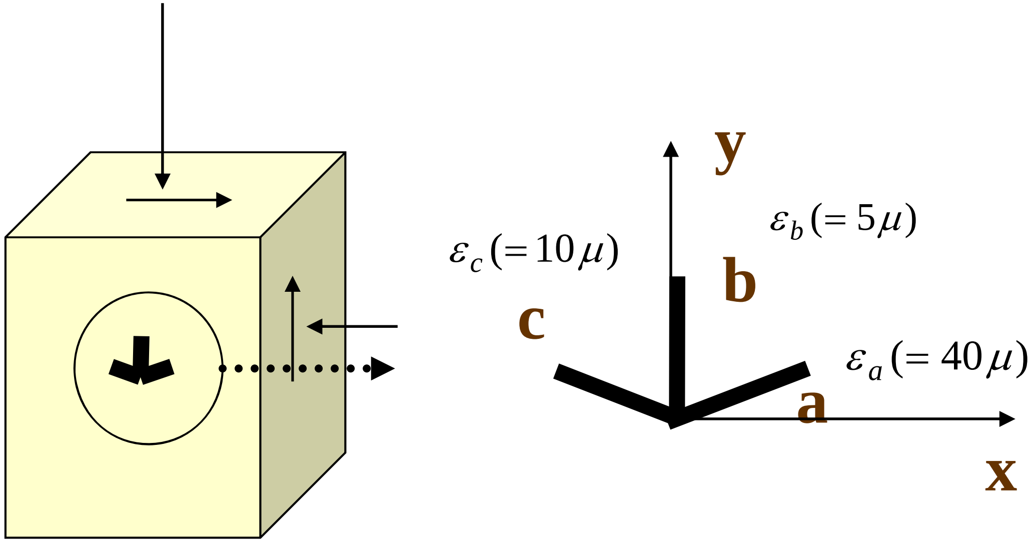
$$\varepsilon(\theta) = \varepsilon_{x'} = \begin{pmatrix} \cos^2\theta & \sin^2\theta & \frac{1}{2}\cdot\sin 2\theta \end{pmatrix} \cdot \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

$$\begin{Bmatrix} \varepsilon(\theta_a) \\ \varepsilon(\theta_b) \\ \varepsilon(\theta_c) \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_a & \sin^2\theta_a & \frac{1}{2}\cdot\sin 2\theta_a \\ \cos^2\theta_b & \sin^2\theta_b & \frac{1}{2}\cdot\sin 2\theta_b \\ \cos^2\theta_c & \sin^2\theta_c & \frac{1}{2}\cdot\sin 2\theta_c \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

三個未知數，三個方程式！

HW7

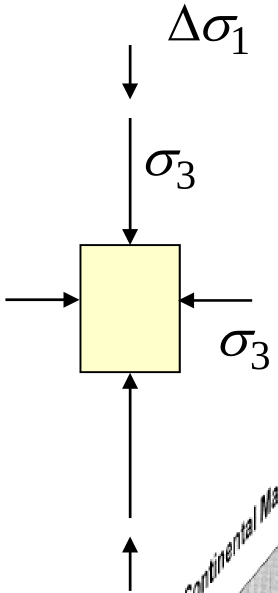
- ***1. Determine the two dimensional strain tensor*** $\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$

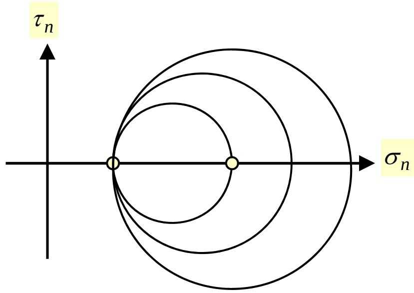


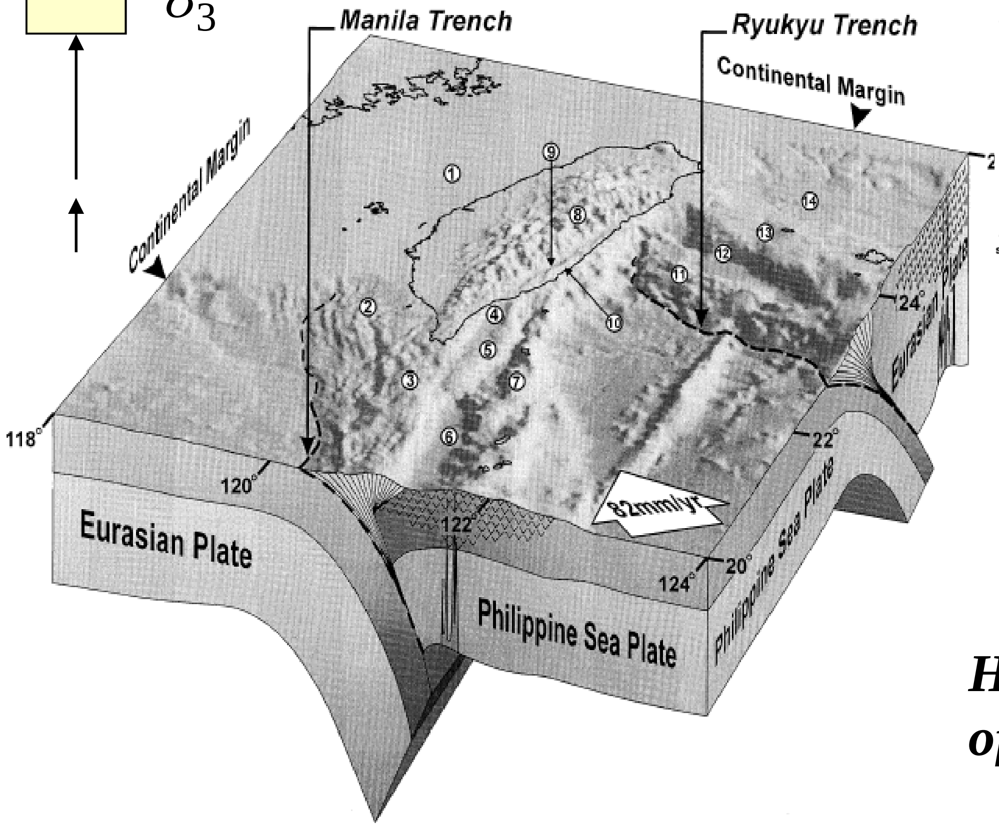
Strain gage a,b,c
分別與水平夾

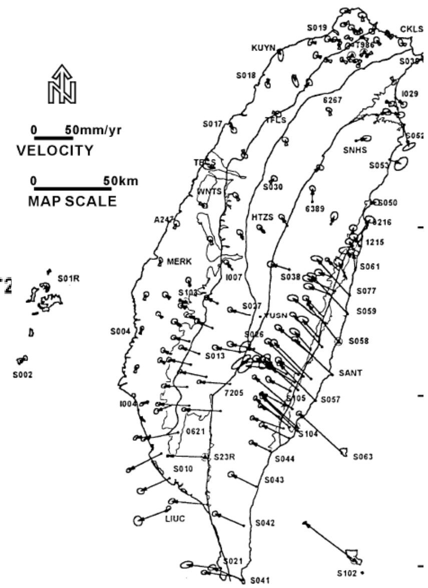
$\theta_a, \theta_b, \theta_c (= 30^o, 90^o, 150^o)$

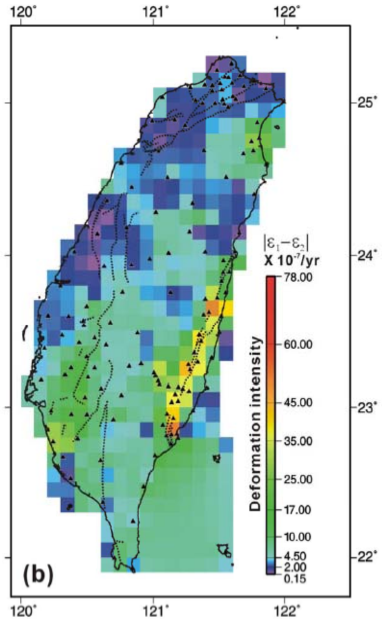
- ***2. Determine the principal strains and principal directions***







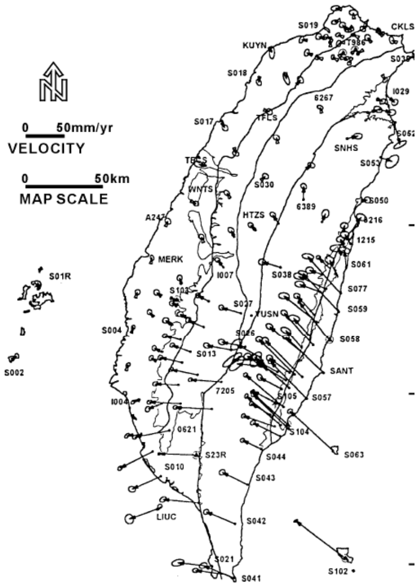


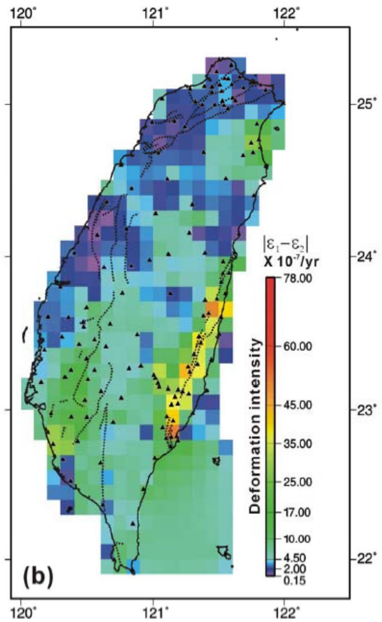


How to determine the strain tensor of a displacement field ?

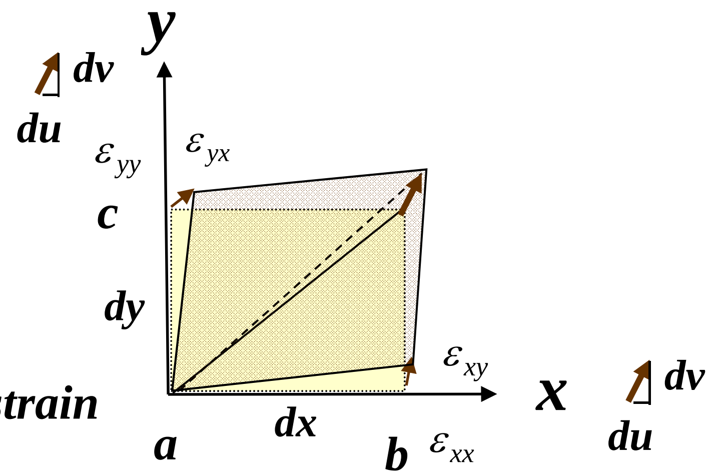
- 菲律賓海板塊與歐亞大陸板塊聚合速率~8cm/year

Strain-displacement relationships





How to determine the strain tensor of a displacement field u(x,y); v(x,y) ?



For small strain

$$\varepsilon_{xx} = \frac{du}{dx}, \varepsilon_{yy} = \frac{dv}{dy}, \gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = \frac{dv}{dx} + \frac{du}{dy}$$

0 year	***a(0,0);***	***b(10000m,0);***	***c(0,10000m)***
5 years later	***a(0,0);***	***b(10001m,1m);***	***c(1m,10001m)***
0-5 year	***u(0,0)=0;*** ***v(0,0)=0;***	***u(10000,0)=1m;*** ***v(10000,0)=1m;***	***u(0,10000)=1m*** ***v(0,10000)=1m***

$$\varepsilon_{xx} = 0.01\%, \varepsilon_{yy} = 0.01\%, \gamma_{xy} = 0.02\%$$

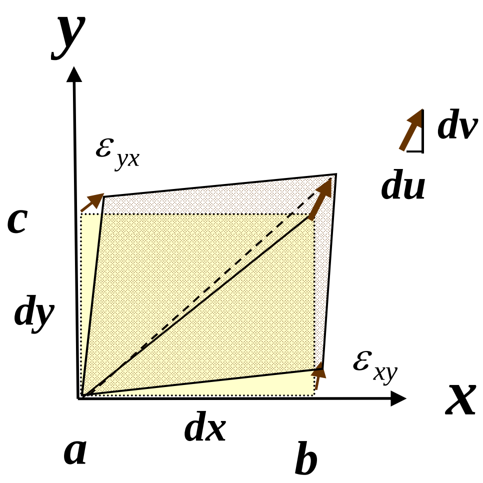
For uniform strain field :

u(x,y)=Ax+By+C
v(x,y)=Dx+Ey+F

C=F=0; A=B=D=E=0.0001

u(x,y)=0.0001(x+y)
v(x,y)=0.0001(x+y)

Strain-displacement relationships



In two dimensional

$$\varepsilon_{xx} = \frac{du}{dx}, \varepsilon_{yy} = \frac{dv}{dy}, \gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = \frac{du}{dy} + \frac{dv}{dx}$$

In three dimensional

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \varepsilon_{zz} = \frac{\partial w}{\partial z},$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

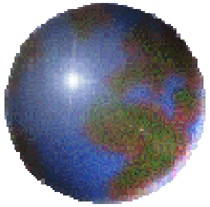
$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

In tensor form

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

此處說明無 **rigid body rotation** *的情況*
考慮剛體旋轉請自行讀一讀 **Section 2.7**



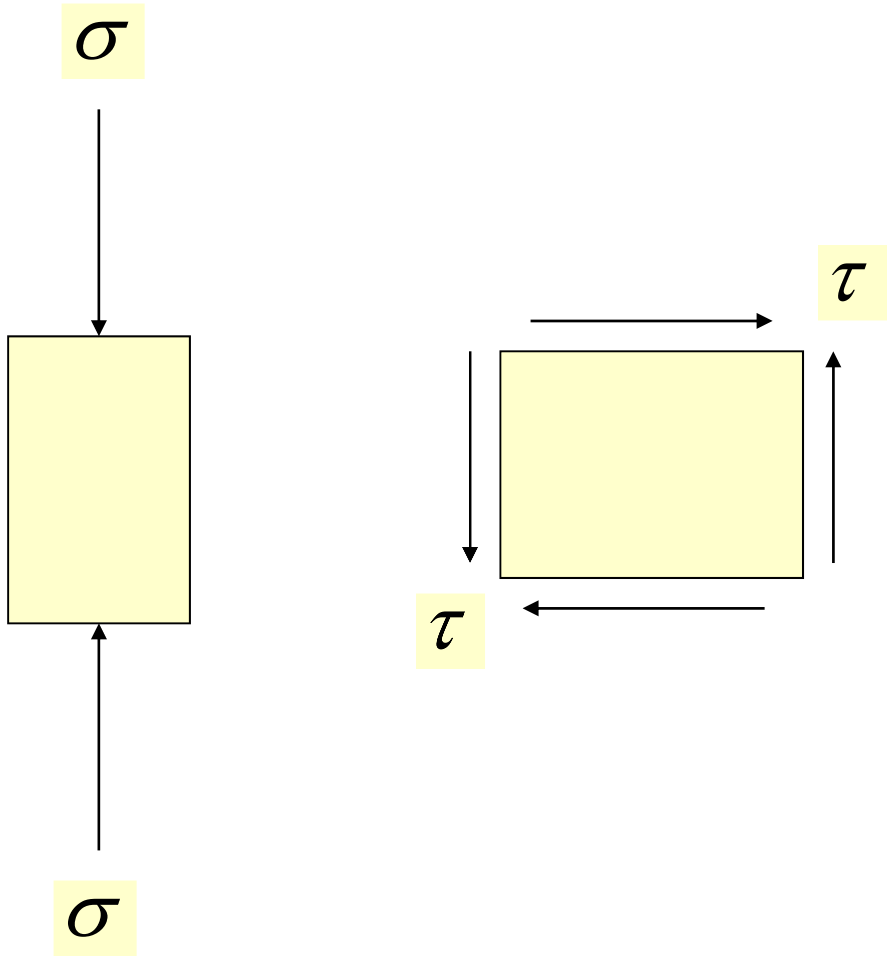
1 Stress and infinitesimal strain

Topic 1 Stress

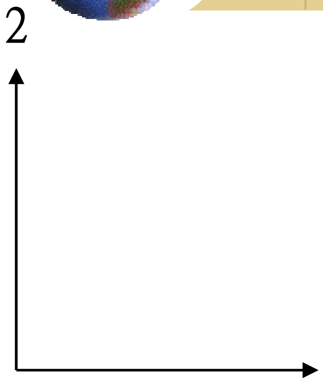
Topic 2 Strain

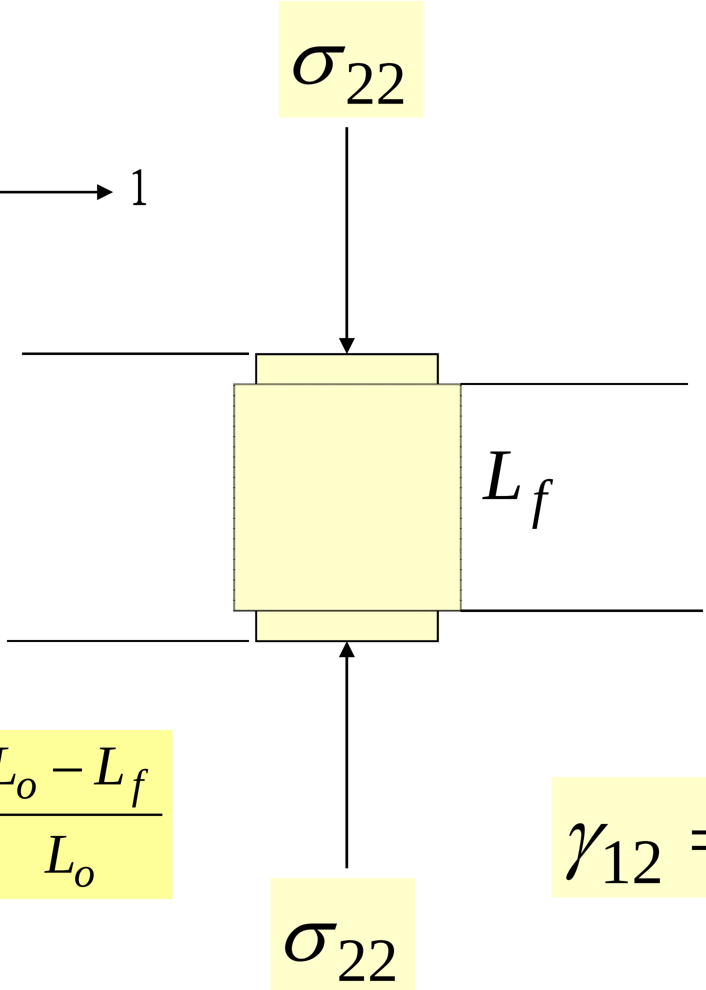
Topic 3 Elastic constants

Stress-strain relationship



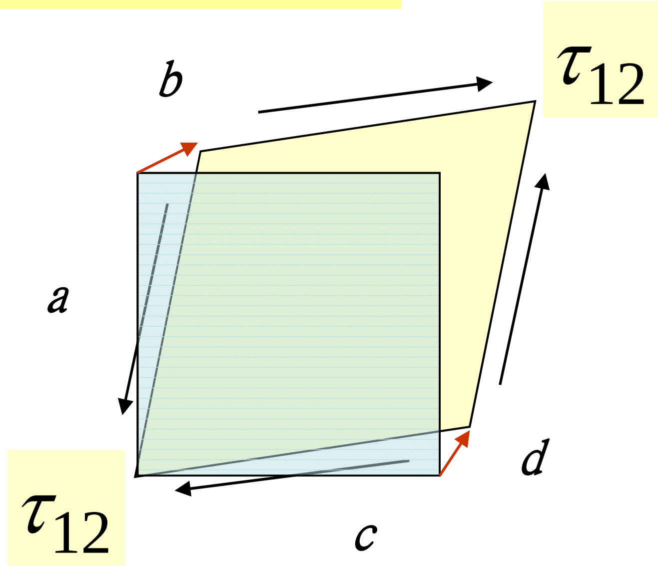
Stress-strain relationship





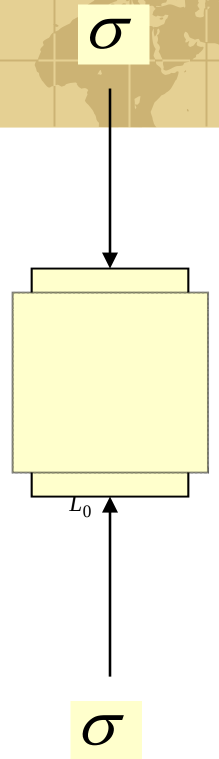
$$\varepsilon_{22} = \frac{L_o - L_f}{L_o}$$

$$\varepsilon_{21} = \frac{b}{a} = \sin\theta = \theta$$

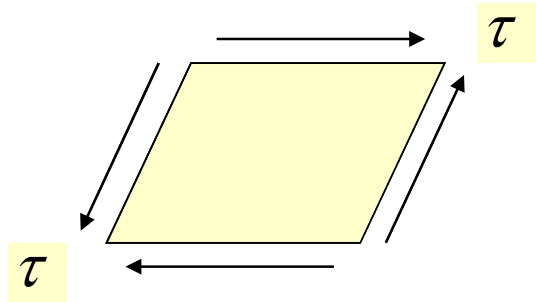


$$\gamma_{12} = \varepsilon_{21} + \varepsilon_{21}$$

$$\varepsilon_{12} = \frac{d}{c} = \sin\theta = \theta$$



$$\varepsilon = \frac{L_o - L_f}{L_o}$$



$$\gamma_{12} = \varepsilon_{12} + \varepsilon_{21}$$

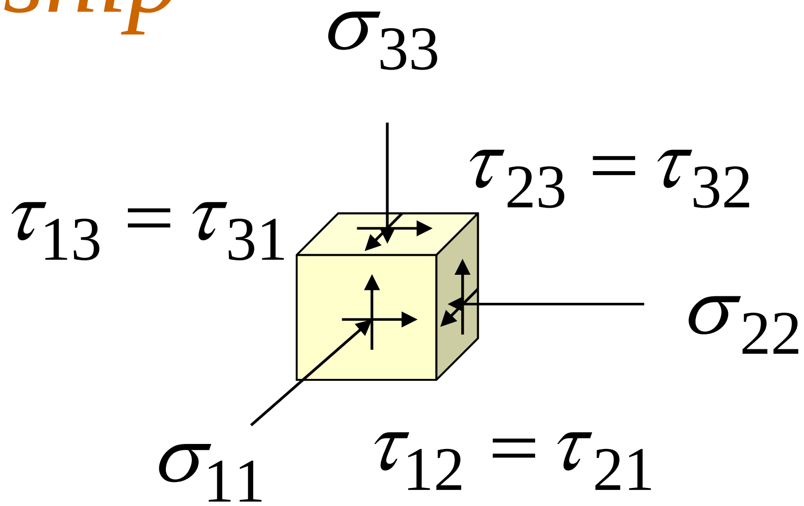
$$\sigma_{11};\ \sigma_{22};\ \sigma_{33} \qquad \tau_{12},\tau_{21};\ \tau_{13},\tau_{31};\ \tau_{23},\tau_{32}$$

$$\varepsilon_{11};\ \varepsilon_{22};\ \varepsilon_{33} \qquad \gamma_{12};\ \gamma_{13};\ \gamma_{23}$$

Stress-strain relationship

- Stress is a tensor
- Strain is a tensor

$$\begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix}$$



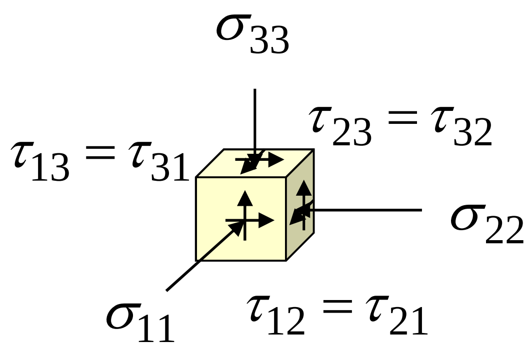
$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

Stress-strain relationship

- Stress and strain tensor are symmetry

$$\begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix}$$

$$\tau_{12} = \tau_{21};\ \tau_{13} = \tau_{31};\ \tau_{23} = \tau_{32}$$



$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

$$\varepsilon_{12} = \varepsilon_{21};\ \varepsilon_{13} = \varepsilon_{31};\ \varepsilon_{23} = \varepsilon_{32}$$

Linear-elastic materials

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \times \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix}$$

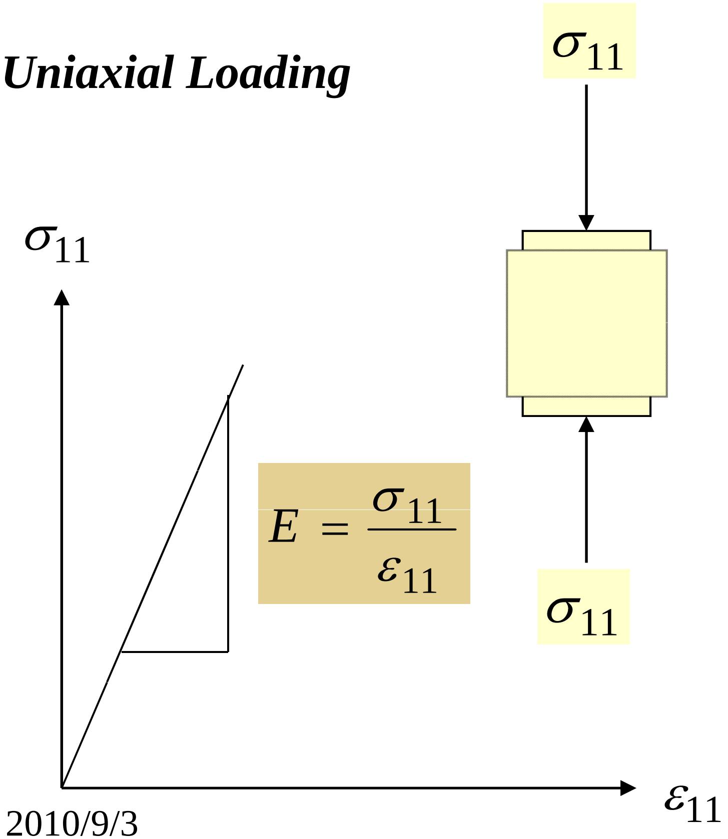
Elastic compliance matrix

How many independent elastic constant ?

Elastic constants (Young's modulus, shear modulus)

- IF $\sigma_{22} = \sigma_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0$

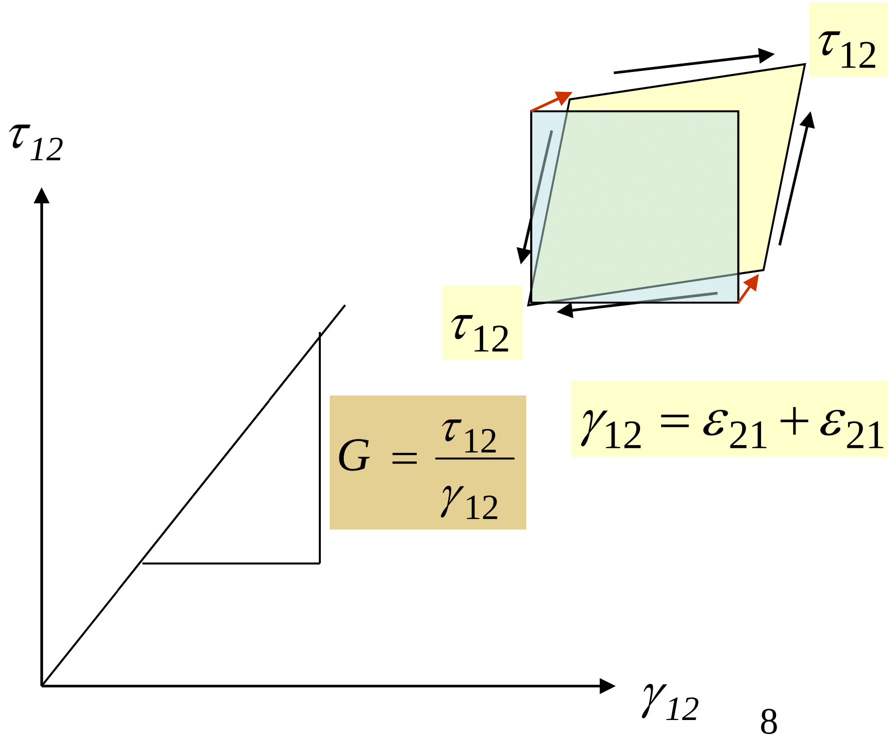
Uniaxial Loading



$$E = \frac{\sigma_{11}}{\varepsilon_{11}}$$

- IF $\sigma_{22} = \sigma_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0$

Simple shear Loading

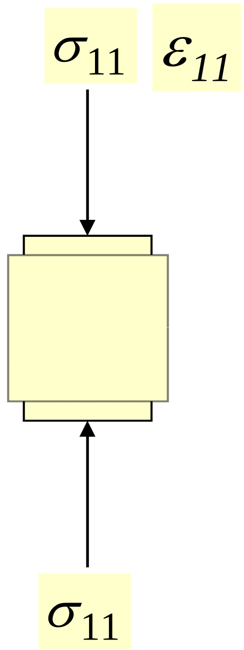


$$G = \frac{\tau_{12}}{\gamma_{12}}$$

$$\gamma_{12} = \varepsilon_{21} + \varepsilon_{21}$$

Elastic constants (Poisson's ratio)

$$E = \frac{\sigma_{11}}{\varepsilon_{11}} \qquad G = \frac{\tau_{12}}{\gamma_{12}}$$



$$\varepsilon_{33} = -\nu \cdot \varepsilon_{11}$$

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}}$$

Linear-elastic, isotropic materials

$$E = \frac{\sigma_{11}}{\varepsilon_{11}} \qquad G = \frac{\tau_{12}}{\gamma_{12}}$$

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}}$$

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{23}\end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \times \begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \tau_{12}\\ \tau_{13}\\ \tau_{23}\end{bmatrix}$$

$$\varepsilon_{11} = +\frac{1}{E}\cdot\sigma_{11} - \frac{\nu}{E}\cdot\sigma_{22} - \frac{\nu}{E}\cdot\sigma_{33}$$

$$\varepsilon_{22} = -\frac{\nu}{E}\cdot\sigma_{11} + \frac{1}{E}\cdot\sigma_{22} - \frac{\nu}{E}\cdot\sigma_{33}$$

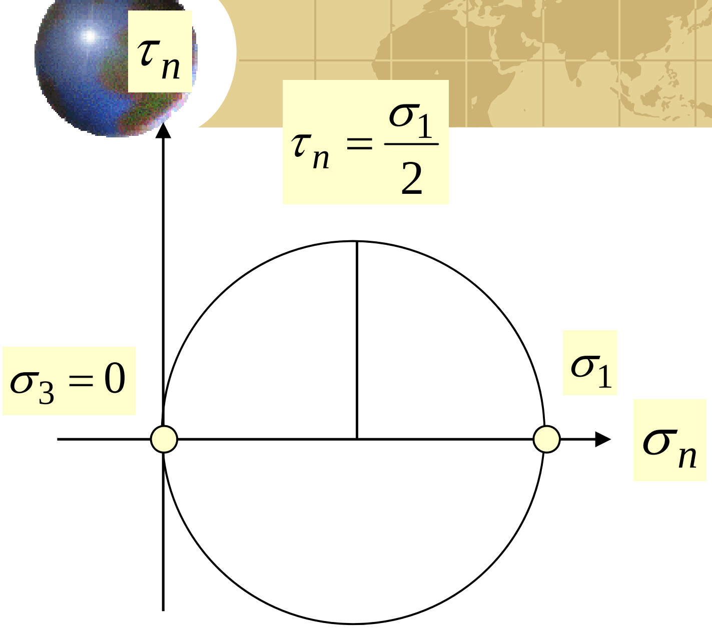
$$\varepsilon_{33} = -\frac{\nu}{E}\cdot\sigma_{11} - \frac{\nu}{E}\cdot\sigma_{22} + \frac{1}{E}\cdot\sigma_{33}$$

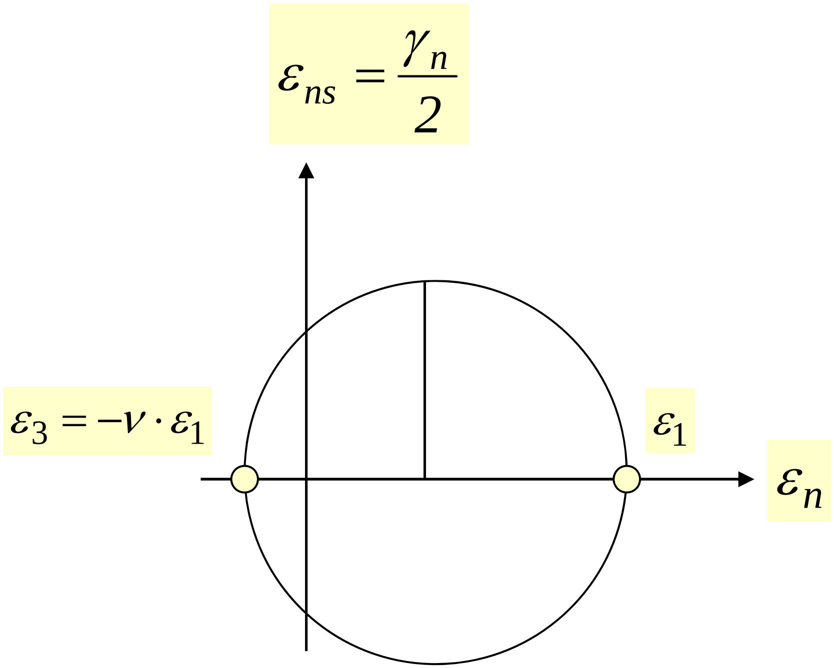
$$\gamma_{12} = \frac{1}{G}\cdot\tau_{12}$$

$$\gamma_{13} = \frac{1}{G}\cdot\tau_{13}$$

$$\gamma_{23} = \frac{1}{G}\cdot\tau_{23}$$

How many independent elastic constant ?





$$\frac{\gamma_n}{2} = \frac{(1+\nu)\cdot\varepsilon_1}{2}$$

$$\frac{\tau_n}{2\cdot G} = \frac{(1+\nu)\cdot\sigma_1}{2\cdot E}$$

$$\because \tau_n = \frac{\sigma_1}{2}$$

$$\frac{1}{2\cdot G} = \frac{(1+\nu)}{E}$$

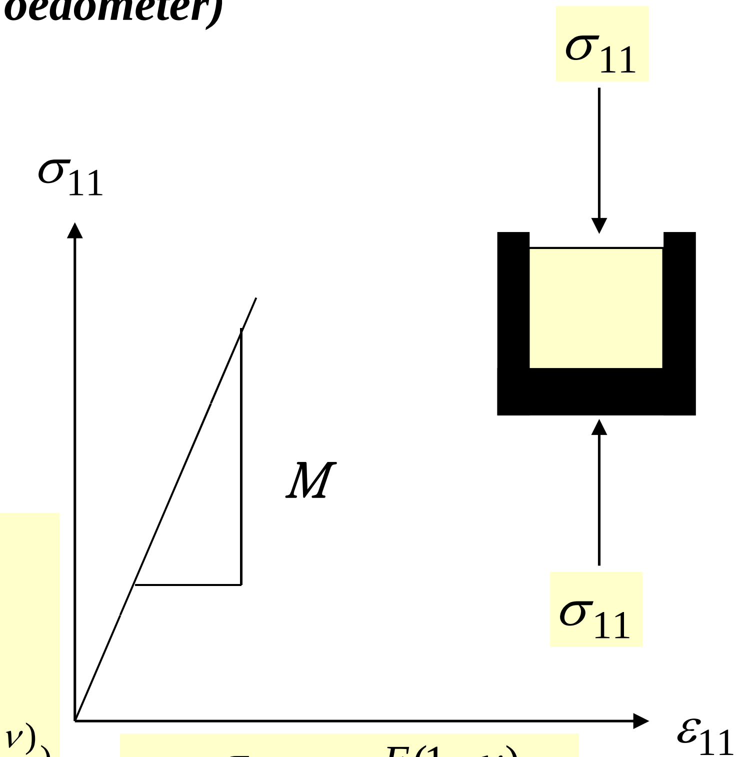
$$G = \frac{E}{2(1+\nu)}$$

Linear-elastic, isotropic materials

$$
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix}
=
\begin{bmatrix}
\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{bmatrix}
\times
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix}
$$

- IF $\varepsilon_{22} = \varepsilon_{33} = \varepsilon_{12} = \gamma_{13} = \gamma_{23} = 0$

Lateral Confined Loading (oedometer)



$$\varepsilon_{22} = -\frac{\nu}{E}\sigma_{11} + \frac{1}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{33} = 0$$

$$\sigma_{22} = \sigma_{33} = \frac{\nu}{1-\nu}\sigma_{11}$$

$$\varepsilon_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{22} + \sigma_{33}) = \frac{\sigma_{11}}{E}(1 - \frac{2\nu^2}{1-\nu}) = \frac{\sigma_{11}}{E}(\frac{(1-2\nu)(1+\nu)}{1-\nu})$$

$$M = \frac{\sigma_{11}}{\varepsilon_{11}} = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}$$